

A NOTE ON TWISTED CONJUGACY AND GENERALIZED BAUMSLAG-SOLITAR GROUPS

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ABSTRACT. A generalized Baumslag-Solitar group is the fundamental group of a graph of groups all of whose vertex and edge groups are infinite cyclic. Levitt proves that any generalized Baumslag-Solitar group has *property* R_∞ , that is, any automorphism has an infinite number of twisted conjugacy classes. We show that any group quasi-isometric to a generalized Baumslag-Solitar group also has property R_∞ . This extends work of the authors proving that any group quasi-isometric to a solvable Baumslag-Solitar $BS(1, n)$ group has property R_∞ , and relies on the classification of generalized Baumslag-Solitar groups given by Whyte.

1. INTRODUCTION

We say that a group G has *property* R_∞ if any automorphism φ of G has an infinite number of twisted conjugacy classes. Two elements $g_1, g_2 \in G$ are φ -twisted conjugate if there is an $h \in G$ so that $hg_1\varphi(h)^{-1} = g_2$. The study of the finiteness of the number of twisted conjugacy classes arises in Nielsen fixed point theory. For example, for each $n \geq 5$, there is a compact n -dimensional nilmanifold M^n whose fundamental group has property R_∞ [GW3]. As a consequence, *every* homeomorphism of such a manifold M^n is isotopic to a fixed point free homeomorphism. For more details and background on fixed point theory, see [B] or [J].

Recently, several authors have studied property R_∞ from a geometric perspective, where the word geometric has a variety of interpretations. It is proven in both [FG2, TWh] that a group which has a non-elementary action by isometries on a Gromov hyperbolic space has property R_∞ , where the action is fundamental to understanding the twisted conjugacy classes. Recently, the authors have given a proof that the lamplighter groups $L_n = \mathbb{Z}_n \wr \mathbb{Z}$ have property R_∞ iff $(n, 6) = 1$, originally proven in [GW1], using mainly the geometry of the Cayley graph of these groups. Namely, the geometry of the Diestel-Leader graph $DL(n, n)$ combined with recent results of Eskin, Fisher and Whyte [EFW1, EFW2] provides a geometric interpretation for the twisted conjugacy classes. [TWo2]

A natural question to ask is whether property R_∞ is *geometric*, that is, invariant under quasi-isometry. It is shown in [FG1] that the Baumslag-Solitar groups $BS(m, n)$ (excepting $BS(1, 1)$) have property R_∞ , and in [TWo1] that any group quasi-isometric to $BS(1, n)$ also has the property. The analogous results are shown in [TWo1] for the solvable generalization Γ of $BS(1, n)$ given by the short exact sequence

$$1 \rightarrow \mathbb{Z}\left[\frac{1}{n}\right] \rightarrow \Gamma \rightarrow \mathbb{Z}^k \rightarrow 1$$

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and any group quasi-isometric to Γ . However, property R_∞ is not in general a quasi-isometry invariant. Let $A, B \in GL(2, \mathbb{Z})$ be matrices whose traces have absolute value at least two. Then $\mathbb{Z}^2 \rtimes_A \mathbb{Z}$ and $\mathbb{Z}^2 \rtimes_B \mathbb{Z}$ are always quasi-isometric, as they are both cocompact lattices in Sol , but may not both have property R_∞ [GW2].

In this note we prove the following theorem about groups quasi-isometric to generalized Baumslag-Solitar groups, extending the result of Levitt [L] as well as the results of [TWo1]. A generalized Baumslag-Solitar group is the fundamental group of a graph of groups all of whose vertex and edge groups are infinite cyclic.

Theorem 3.1. *Let G be a finitely generated group quasi-isometric to a non-elementary generalized Baumslag-Solitar group. Then G has property R_∞ .*

Our proofs rely on a result of Whyte [W] stating that any group G quasi-isometric to a generalized Baumslag-Solitar group has one of three forms: either G is $BS(1, n)$ and the result is proven in [TWo1], G is virtually $F_n \times \mathbb{Z}$ or G is the fundamental group of a graph of groups all of whose vertex and edge groups are *virtually* infinite cyclic.

In the second case above, we rely on work of Sela [S] at a crucial step to guarantee that the quotient group we are considering, which is a non-elementary Gromov hyperbolic group, is Hopfian. This quotient group is obtained by considering a *quasi-action*, as an action may not exist, of the group on the product of a tree with the real line.

The third case splits into two subcases depending on whether G is unimodular. If it is not, then Levitt's proof that any generalized Baumslag-Solitar group has property R_∞ applies verbatim to G . When a generalized Baumslag-Solitar group is unimodular, Levitt studies central elements which are necessarily elliptic, and concludes that the center of the group is either trivial or infinite cyclic. When G is quasi-isometric to a generalized Baumslag-Solitar group, we can only conclude that the center is *virtually* \mathbb{Z} or else the normal closure of the torsion elements of G . We then show that this cannot occur, that is, in this case G cannot be unimodular.

Thus the complete proof of Theorem 3.1 combines a number of existing techniques and theorems due to Levitt [L], Kleiner and Leeb [KL], Whyte [W] and the authors [TWo1], and adds a new and interesting class to the list of groups for which property R_∞ is invariant under quasi-isometry. We note that part of the main theorem could also be proven using a result of Rieffel [R].

2. BACKGROUND ON TWISTED CONJUGACY AND QUASI-ISOMETRIES

2.1. Twisted conjugacy. Let $\varphi : \pi \rightarrow \pi$ be a group endomorphism. We consider the action of π on π given by $\sigma \cdot \alpha \mapsto \sigma \alpha \varphi(\sigma)^{-1}$ for $\sigma, \alpha \in \pi$. The orbits of this action are the *Reidemeister classes* of φ or the φ -twisted conjugacy classes. Denote by $R(\varphi)$ the cardinality of the set $\mathcal{R}(\varphi)$ of φ -twisted conjugacy classes. This number $R(\varphi)$ is called the *Reidemeister number* of φ . When φ is the identity, $\mathcal{R}(\varphi)$ is the set of conjugacy classes of elements of π , and $R(\varphi)$ is simply the number of conjugacy classes.

We say that a group G has *property R_∞* if, for any $\varphi \in \text{Aut}(G)$, we have $R(\varphi) = \infty$. The main technique we use for computing $R(\varphi)$ is as follows. We consider groups which can be expressed as group extensions, for example $1 \rightarrow A \rightarrow B \rightarrow C \rightarrow 1$. Suppose that an automorphism $\varphi \in \text{Aut}(B)$ induces the following commutative diagram, where the vertical arrows are group homomorphisms, that is, $\varphi|_A = \varphi'$ and $\overline{\varphi}$ is the quotient map induced by φ on C :

$$(1) \quad \begin{array}{ccccccc} 1 & \longrightarrow & A & \xrightarrow{i} & B & \xrightarrow{p} & C \longrightarrow 1 \\ & & \varphi' \downarrow & & \varphi \downarrow & & \overline{\varphi} \downarrow \\ 1 & \longrightarrow & A & \xrightarrow{i} & B & \xrightarrow{p} & C \longrightarrow 1 \end{array}$$

Then we obtain a short exact sequence of sets and corresponding functions \hat{i} and \hat{p} :

$$\mathcal{R}(\varphi') \xrightarrow{\hat{i}} \mathcal{R}(\varphi) \xrightarrow{\hat{p}} \mathcal{R}(\overline{\varphi}) \rightarrow 1$$

where if $\bar{1}$ is the identity element in C , we have $\hat{i}(\mathcal{R}(\varphi')) = \hat{p}^{-1}([\bar{1}])$, and \hat{p} is onto. To ensure that both φ' and $\overline{\varphi}$ are both automorphisms, we need the following lemma. Recall that a group is *Hopfian* if every epimorphism is an automorphism. The following lemma is proven in [TWo2].

Lemma 2.1. *If C is Hopfian, then $\varphi' \in \text{Aut}(A)$ and $\overline{\varphi} \in \text{Aut}(C)$.*

The following result is straightforward and follows from more general results discussed in [Wo].

Lemma 2.2. *Given the commutative diagram labeled (1) above,*

- (1) *if $R(\overline{\varphi}) = \infty$ then $R(\varphi) = \infty$, and*
- (2) *if C is finite and $R(\varphi) = \infty$ then $R(\varphi') = \infty$.*

2.2. Quasi-isometries and quasi-actions. A quasi-isometry is a map between metric spaces which distorts distance by a uniformly bounded amount, defined precisely as follows.

Definition 2.3. *Let X and Y be metric spaces. A map $f : X \rightarrow Y$ is a (K, C) -quasi-isometry, for $K \geq 1$ and $C \geq 0$ if*

- (1) *$\frac{1}{K}d_X(x_1, x_2) - C \leq d_Y(f(x_1), f(x_2)) \leq Kd_X(x_1, x_2) + C$ for all $x_1, x_2 \in X$.*
- (2) *For some constant C' , we have $\text{Nhbdc}_{C'}(f(X)) = Y$.*

There is a notion of *coarse inverse* for a quasi-isometry; quasi-isometries f and g are coarse inverses if both compositions $f \circ g$ and $g \circ f$ are a bounded distance from the identity. Without loss of generality we may assume that g and f share the same quasi-isometry constants.

The set of all self quasi-isometries of a finitely generated group G is denoted $QIMap(G)$. We form equivalence classes consisting of all quasi-isometries of G which differ by a uniformly bounded amount. This set of equivalence classes is called the *quasi-isometry group* of G and denoted $QI(G)$.

If G is a finitely generated group and X is a proper geodesic metric space, we define a *quasi-action* of G on X to be a map $\Psi : G \rightarrow QIMap(X)$ satisfying the following properties for some constants $K \geq 1$ and $C \geq 0$.

- (1) For each $g \in G$, the element $\Psi(g)$ is a (K, C) -quasi-isometry of X .
- (2) $\Psi(Id)$ is a uniformly bounded distance from the identity, that is, $\Psi(Id)$ is the identity in $QI(X)$.
- (3) $\Psi(g)\Psi(h)$ is a uniformly bounded distance from $\Psi(gh)$, for all $g, h \in G$.

It is clear that a quasi-action induces a homomorphism from G into $QI(X)$. It is important to note that all quasi-isometries $\Psi(g)$ have the same quasi-isometry constants.

When using Lemma 2.2 to prove that a group G has property R_∞ , one must be able to find characteristic subgroups of G . One such characteristic subgroup of G is the *virtual center* $V(G)$.

This subgroup consists of elements of G whose centralizers have finite index in G , that is, $V(G) = \{g \in G \mid [G : C(g)] < \infty\}$. When G has a quasi-action on a Cayley complex, it follows from Lemma 2.4 below that the virtual center of G is exactly the kernel of this quasi-action. Namely, we use the fact that $g \in G$ moves all points of G a uniformly bounded distance under left multiplication if and only if g has finitely many conjugates, and show that the virtual center consists exactly of those group elements having finitely many conjugates. This gives another characterization of the virtual center of G . By construction, the virtual center is a characteristic subgroup of G .

Lemma 2.4. *Let G be a group generated by a finite set S which has a quasi-action on a Cayley complex X . Then the virtual center of G consists of those elements that move all points of X a uniformly bounded distance B , that is, $d(gx, x) \leq B$ for all $x \in X$, where distance is computed in the word metric with respect to S .*

Proof. To prove this lemma, we use the fact that $g \in G$ moves all points of G a uniformly bounded distance under left multiplication if and only if g has finitely many conjugates. This is true because there are a finite number of elements of G in the ball of radius B in any Cayley graph of G with respect to a finite generating set.

We first show that if $g \in V(G)$, then g has finitely many conjugates in G . Fix an element g in the virtual center $V(G)$. Since $[G : C(g)] < \infty$, we can write G as a disjoint union of the cosets of $C = C(g)$, namely $C, a_1C, a_2C, \dots, a_nC$ for some $a_1, a_2, \dots, a_n \in G$.

Take any two elements from the coset a_1C , say a_1c_1 and a_1c_2 where $c_1, c_2 \in C$. When we conjugate g by these two elements we see that

$$(a_1c_1)g(a_1c_1)^{-1} = a_1(c_1gc_1^{-1})a_1^{-1} = a_1ga_1^{-1},$$

where the last equality follows because $c_1 \in C(g)$, and

$$(a_1c_2)g(a_1c_2)^{-1} = a_1(c_2gc_2^{-1})a_1^{-1} = a_1ga_1^{-1}.$$

Thus we see that the only possible conjugates of g have the values $a_iga_1^{-1}$ for $i = 1, \dots, n$, or g itself.

Now assume that g has finitely many conjugates in G . We will show that the centralizer $C = C(g)$ has finite index in G . Suppose there were infinitely many cosets of C , which we denote a_1C, a_2C, \dots for an infinite sequence $a_1ma_2 \dots \in G$. Consider the conjugates $a_i^{-1}ga_i$. Since g has finitely many conjugates, we know that infinitely many of these must be the same. Suppose that $a_1^{-1}ga_1 = a_2^{-1}ga_2$. This is equivalent to $(a_2^{-1}a_1)^{-1}g(a_2^{-1}a_1) = g$, so $a_2^{-1}a_1$ must be in $C(g)$, so as cosets $a_1C = a_2C$. Thus there are a finite number of cosets and C has finite index in G . \square

2.3. Group actions on trees. We assume that G is a finitely generated group acting simplicially on a tree T , that is, this action preserves vertices and edges of T , without inversions. Moreover, we require this action to be *minimal*, meaning that there is no proper invariant subtree under this action.

An element $g \in G$ is called *elliptic* if g fixes a vertex in T , and *hyperbolic* otherwise. These properties are best defined in terms of translation length, as follows. View T as a metric space by assigning each edge length one. Define the *translation length* l_g of an element $g \in G$ acting on T to be the minimum distance between a vertex $x \in V(T)$ and its image $g \cdot x$, that is,

$$l_g = \min_{x \in V(T)} d(x, g \cdot x).$$

If $l_g = 0$ then we say that g is *elliptic*, and g has a fixed point when acting on T . Otherwise g is *hyperbolic*, and T has a g -invariant linear subtree A_g , called the axis of g , consisting of the following

set of points:

$$A_g = \{x \in T \mid d(x, g \cdot x) = l_g\}.$$

The notion of commensurability will also play a role in the proofs below. There are several standard definitions of commensurability. Two groups are abstractly commensurable if they have isomorphic finite index subgroups. Two subgroups H_1 and H_2 of a given group G are commensurable if their intersection has finite index in both subgroups. We say that two elements $g, h \in G$ are commensurable if the subgroups they generate are commensurable in G .

When G is a generalized Baumslag-Solitar group, all elliptic elements have infinite order and are commensurable. For the groups we consider, in which all vertex and edge stabilizers are only virtually infinite cyclic, all infinite order elliptic elements are still commensurable, and the same is true for any finite order elliptic elements. In general, the properties of being elliptic or hyperbolic and having finite or infinite order are preserved under both conjugation and commensurability. This may not be true when considering twisted conjugacy, however. In the case of the infinite dihedral group D_∞ , the order two elliptic elements are φ -twisted conjugate to the infinite order elliptic elements where $\varphi : \mathbb{Z} \rightarrow \mathbb{Z}$ is given by $\varphi(x) = -x$.

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Below we prove that any group quasi-isometric to a non-elementary generalized Baumslag-Solitar group has property R_∞ . A generalized Baumslag-Solitar group is a finitely generated group which acts on a tree with all edge and vertex stabilizers infinite cyclic, that is, the fundamental group of a graph of groups all of whose vertex and edge groups are infinite cyclic. A group is non-elementary if it is not virtually cyclic.

According to [W], Theorem 0.1, any group Γ which is a generalized Baumslag-Solitar group has one of three forms:

- (1) $\Gamma = BS(1, n)$ for some $n > 1$
- (2) Γ is virtually $F_n \times \mathbb{Z}$
- (3) Γ is quasi-isometric to $BS(2, 3)$.

We will consider these three cases in the proof below. We now prove the following theorem.

Theorem 3.1. *Let G be a finitely generated group quasi-isometric to a non-elementary generalized Baumslag-Solitar group. Then G has property R_∞ .*

Proof. Case 1. If G is quasi-isometric to $\Gamma = BS(1, n)$ for some $n > 1$ then it is proven in [TWo1] that $R(\varphi) = \infty$ for all $\varphi \in \text{Aut}(G)$.

Case 2. If G is quasi-isometric to Γ , which is virtually $F_n \times \mathbb{Z}$, then G itself is quasi-isometric to $F_n \times \mathbb{Z}$. The geometric model of this group, also called the Cayley complex, which is quasi-isometric to the group, is then the product of a tree T with \mathbb{R} . In particular, since $n \geq 2$, the tree T is not a line.

It follows from [KL], Theorem 1.1, that G fits into a short exact sequence

$$1 \rightarrow H \rightarrow G \rightarrow L \rightarrow 1$$

where H is virtually \mathbb{Z} and L is a uniform lattice in the isometry group of T , and thus Gromov hyperbolic.

The short exact sequence above is obtained by constructing a quasi-action of G on $T \times \mathbb{R}$ so that L is the image and H is the kernel of this quasi-action. It then follows from Lemma 2.4 that H is the virtual center of G , and thus characteristic under any group automorphism $\varphi \in \text{Aut}(G)$. This allows us to induce a surjective homomorphism $\bar{\varphi} : L \rightarrow L$. Since L is a non-elementary Gromov hyperbolic group, it is Hopfian by [S] and so Lemma 2.1 implies that $\bar{\varphi} \in \text{Aut}(L)$. It follows from [LL, F] that $R(\bar{\varphi}) = \infty$, and from Lemma 2.2 that $R(\varphi) = \infty$ as well.

We note that this case could also be proven using the main result of [R], which is a special case of Theorem 1.1 of [KL].

Case 3. If G is quasi-isometric to Γ , which is quasi-isometric to $BS(2, 3)$, then G itself is quasi-isometric to $BS(2, 3)$. We quote the following theorem of Whyte which describes groups quasi-isometric to $BS(2, 3)$.

Theorem 3.2 ([W], Theorem 5.1). *Let Γ be a finitely generated group. Then Γ is quasi-isometric to $BS(2, 3)$ iff Γ is the fundamental group of a graph of virtual \mathbb{Z} 's which is neither commensurable to $F_n \times \mathbb{Z}$ nor virtually solvable.*

Thus our group G is the fundamental group of a graph of groups all of whose vertex and edge groups are virtually infinite cyclic. Let T be the tree on which G acts with stabilizers which are virtually infinite cyclic. The elliptic elements of G fall into two classes: those with finite order, and those with infinite order. Using the fact that all infinite order elliptic elements in G are commensurable, we can define the modular homomorphism $\Delta : G \rightarrow \mathbb{Q}^*$ as follows.

Fix an infinite order elliptic element α , and let g be any element of G . Since $g\alpha g^{-1}$ will be an infinite order elliptic element, and thus commensurable to α , we see that there is a relator of the form $g\alpha^p g^{-1} = \alpha^q$ for some $p, q \in \mathbb{Z} - \{0\}$. Define $\Delta(g) = \frac{p}{q}$, which is well defined because all infinite order elliptic elements are commensurable. Since automorphisms of G preserve both finite or infinite order and the type of element (elliptic or hyperbolic), we see that $\Delta \circ \psi = \Delta$ for any $\psi \in \text{Aut}(G)$.

We say that a group G is *unimodular* if the image of Δ is contained in $\{\pm 1\}$. We now quote Levitt's proof from [L], Proposition 2.7 that when G is not unimodular, $R(\varphi) = \infty$. Namely, the image of Δ is infinite, and φ -conjugate elements have the same modulus, so the result follows.

We now suppose that G is unimodular, and adapt the proofs of [L], Propositions 2.5, 2.6 and 2.7. Following Levitt, we note that all central elements in G are elliptic, and that the center $Z(G)$ is contained in the kernel of the action of G on T . In the case of generalized Baumslag-Solitar groups, one can conclude that $Z(G)$ is trivial or infinite cyclic [L]. However, in our case, we can only say that it is either virtually \mathbb{Z} or the normal closure of the torsion elements of G , since $Z(G)$ must be contained in every vertex stabilizer, all of which are virtually \mathbb{Z} .

If $\Delta(G)$ is trivial, we conclude following Levitt's argument that $Z(G)$ is virtually \mathbb{Z} . In this case, we note that $G/Z(G)$ acts on T with finite stabilizers, and thus must be virtually free. We obtain the short exact sequence

$$(2) \quad 1 \rightarrow Z(G) \rightarrow G \rightarrow F \rightarrow 1$$

where F is virtually free. Since we have assumed that T is not a line, we know that F contains a free group F_n with $n \geq 2$ as a subgroup of finite index. The pullback of (2) by the inclusion $F_n \hookrightarrow F$ is an extension G' of $Z(G)$ by F_n , namely

$$(3) \quad 1 \rightarrow Z(G) \rightarrow G' \rightarrow F_n \rightarrow 1$$

Since F_n is free, the sequence splits. When we view the group G' as a finite index subgroup of G , and recall that $Z(G)$ is the center of G , it follows that G' is a direct product, that is, $G' = Z(G) \times F_n$.

From this we conclude that G is virtually $Z(G) \times F_n$. As $Z(G)$ is virtually infinite cyclic, we see that G is virtually $\mathbb{Z} \times F_n$. Following Theorem 3.2, the case $\Delta(G) = 1$ cannot occur.

We now consider the case when $Z(G)$ is the normal closure of the torsion elements of G , which corresponds to the case when $\Delta(G) \subset \{\pm 1\}$. Consider $K = \text{Ker}(\Delta)$ which has index two in G . We note that K is also the fundamental group of a graph of groups all of whose vertex and edge groups are virtually \mathbb{Z} : K acts on the same tree as G , and $[G : K] = 2$ implies that all vertex and edge stabilizers must again be virtually infinite cyclic. Since $\Delta(K)$ is trivial, we conclude as in the case $\Delta(G) = 1$ that this case cannot occur. Hence, in case 3, G cannot be unimodular and the proof is complete. \square

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